Black Hole Astrophysics
Chapter 7.4

All figures extracted from online sources of from the textbook.

Flowchart

Basic properties of the Schwarzschild metric

Coordinate systems

Equation of motion and conserved quantities

Let's throw stuff in!

What does it feel like to orbit a Black Hole?

General motion in Schwarzschild Metric

Horizon Penetrating coordinates

The Schwarzschild Matric

“Sch” means that this metric is describing a Schwarzschild Black Hole.

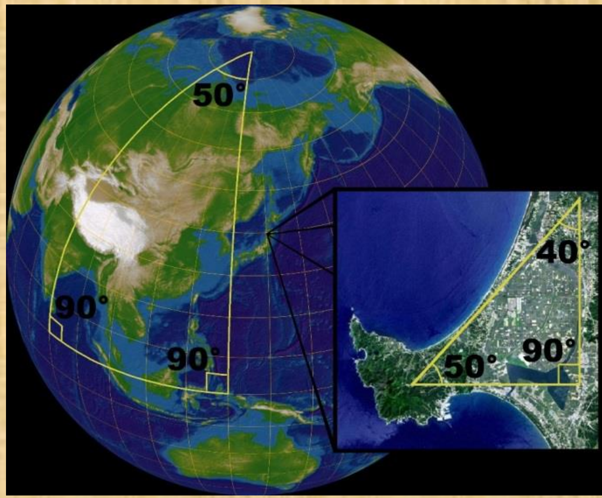
$$\left(g_{\mathrm{SH}}^{\mathrm{Sch}}\right)_{\alpha\beta} = \begin{pmatrix} -c^2\left(1-\frac{r_s}{r}\right) & 0 & 0 & 0 \\ 0 & \dfrac{1}{\left(1-\frac{r_s}{r}\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

Recall: A Schwarzschild Black Hole is a solution of the Einstein Equations assuming that we put a point mass M in free space and then assume that we are in a static coordinate.

The Schwarzschild radius $r_s = \frac{2\mathrm{GM}}{c^2}$

“SH” means that we are in the Schwarzschild-Hilbert coordinate system.

Why bother?
Remember that we are now in curved space, but we can sometimes for convenience still choose a locally flat coordinate to consider the physics. The SH coordinate is just like considering the whole surface of the Earth as a curved surface.



The metric being diagonal also says that relativistic spherical gravity is still a radial r=force.

Some basic properties

Nevertheless, the proper distance between two closely-spaced spheres at r and $r + \Delta r$ can be computed by setting $dt = d\theta = d\phi = 0$ in the Schwarzschild line element, yielding

$$\Delta s \approx \frac{1}{(1 - \frac{r_S}{r})^{1/2}} \Delta r \tag{7.26}$$

where r_S is the Schwarzschild radius

$$\boxed{r_S \equiv \frac{2\,G\,M}{c^2}} \tag{7.27}$$

If we integrate this as a differential, from the horizon outward, we find that the proper radial distance from the *horizon* at r_S to a point at r is given by the complicated expression

$$s = r\left(1 - \frac{r_S}{r}\right)^{1/2} + r_S \ln\left\{\left(\frac{r}{r_S}\right)^{1/2}\left[1 + \left(1 - \frac{r_S}{r}\right)^{1/2}\right]\right\}$$

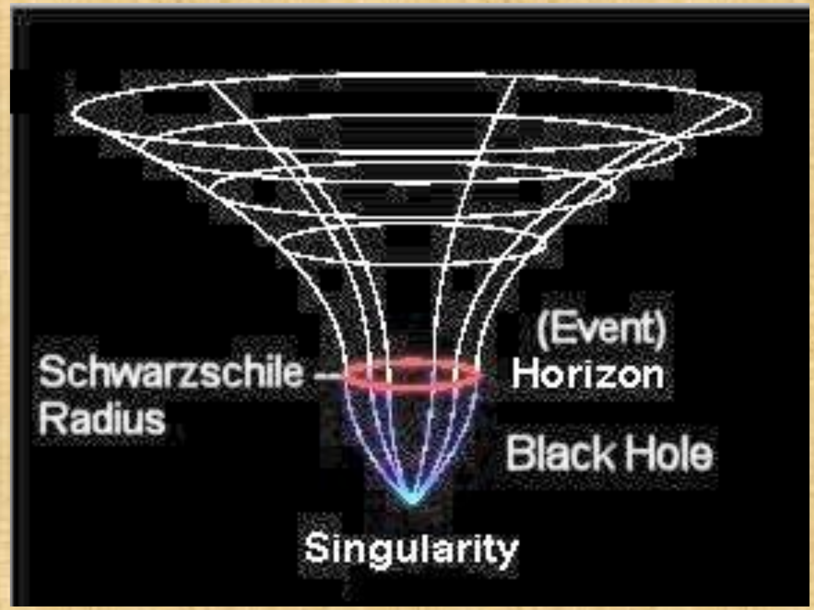
When $r \to r_S$ we have $s = 0$, of course. And as $r \to \infty$, the first term dominates and we have $s \to r$. So, when r is large, it is a good approximation to the proper distance, but when r gets close to r_S, the differential proper distance Δs becomes much larger than Δr itself (equation (7.26)).

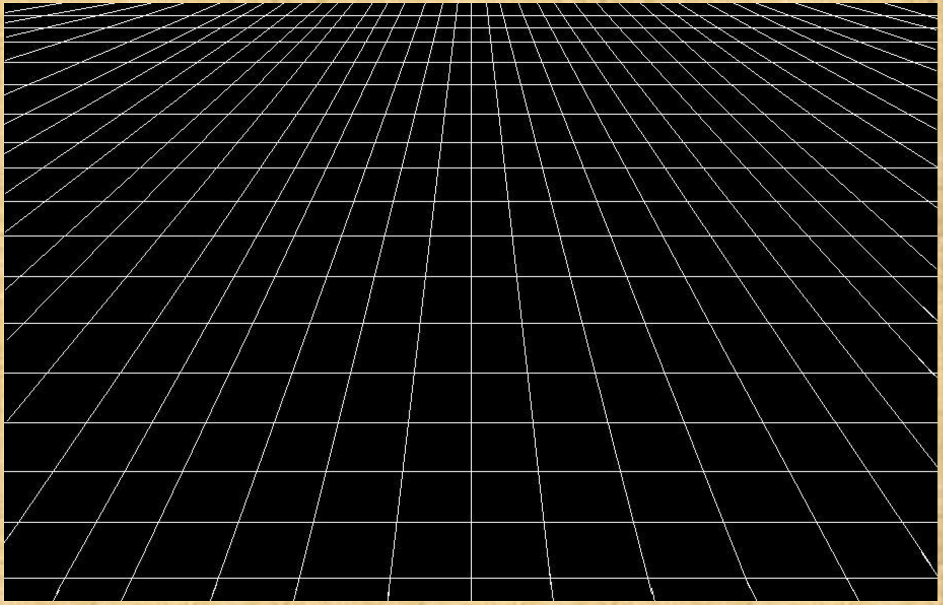
Limits at infinity

$$\left(g_{\mathrm{SH}}^{\mathrm{Sch}}\right)_{\alpha\beta} = \begin{pmatrix} -c^2\left(1-\frac{r_s}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\left(1-\frac{r_s}{r}\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \rightarrow \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

If we take $r \rightarrow \infty$ or $r_s \rightarrow 0$ $(i.e. M \rightarrow 0)$

Reduces to the Minkowski metric!





Passing the horizon

Outside the horizon $r > r_s$ → Inside the horizon $r < r_s$

$$\begin{pmatrix} -c^2\left(1-\frac{r_s}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\left(1-\frac{r_s}{r}\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \rightarrow \begin{pmatrix} c^2\left(\frac{r_s}{r}-1\right) & 0 & 0 & 0 \\ 0 & -\frac{1}{\left(\frac{r_s}{r}-1\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

What's so interesting?
We know that particles can only travel on timelike trajectories, that is, $\mathrm{ds}^2 < 0$.

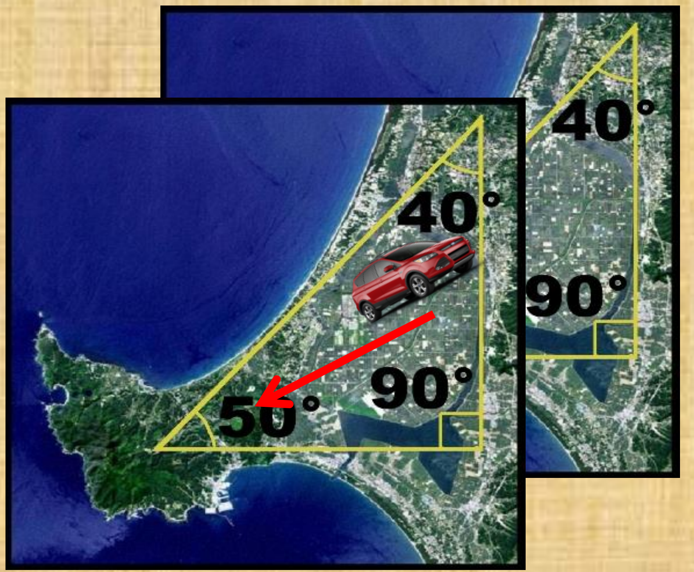
Outside the horizon, g_{tt} is the negative term so we can be on a timelike trajectory if we have $\mathrm{dt} \neq 0, \mathrm{dr} = \mathrm{d\theta} = \mathrm{d\phi} = 0$

Inside the horizon, it is g_{rr} that is negative! So to be on a timelike trajectory, the simplest case would be to have $\mathrm{dr} \neq 0, \mathrm{dt} = \mathrm{d\theta} = \mathrm{d\phi} = 0$

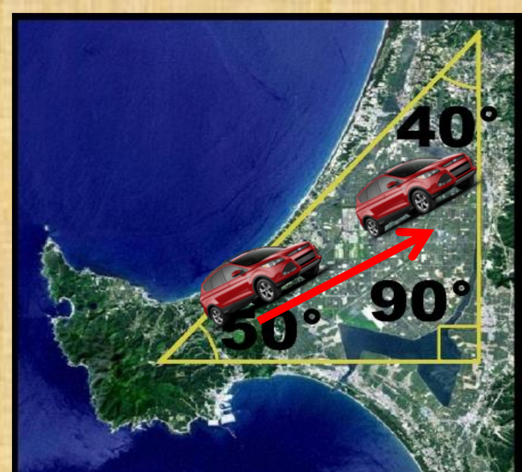
This means that we can only fall toward the BH once we pass the horizon!

Coordinate Systems

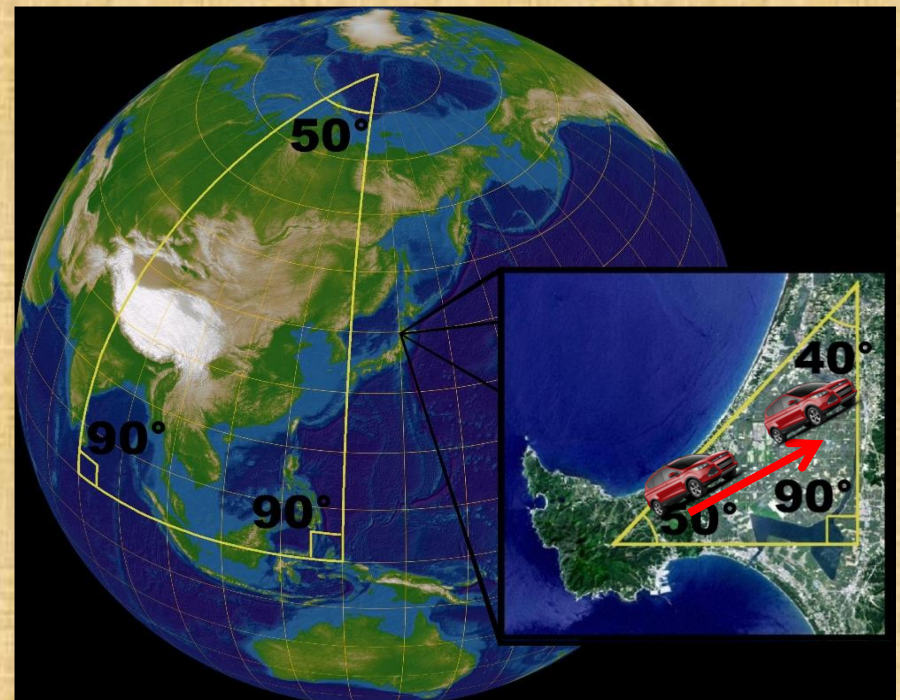
1. The moving body frame (MOV)



2. Fixed local Lorentz frame (FIX)



3. Schwarzschild-Hilbert frame (SH)



The moving body frame (MOV)

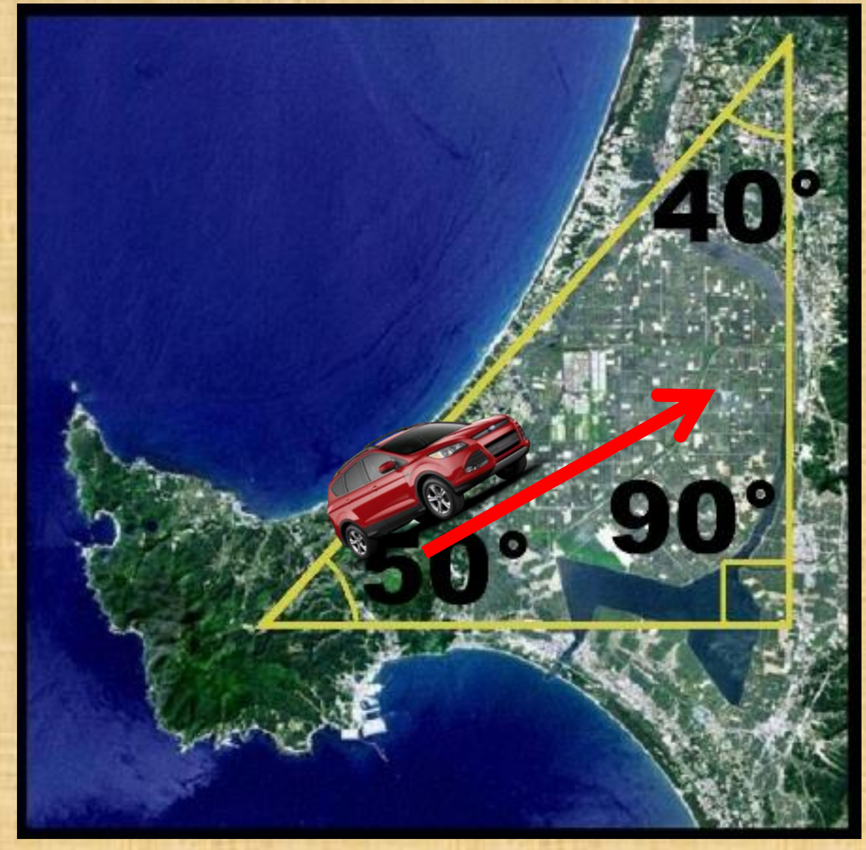
In this frame, we are moving with the object of interest. Since spacetime is locally flat, we have a Minkowski metric in this case

$$\left(g^{\mathrm{Sch}}_{\mathrm{MOV}}\right)_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and by definition the 4-velocity $(U_{\mathrm{MOV}})^{\alpha} = (c, 0,0,0)$

This frame is useful for expressing microphysics, such as gas pressure, temperature, and density, but not motion.

Fixed local Lorentz frame (FIX)



In this frame, we consider some locally flat part of the Schwarzschild spacetime to sit on and watch things fly past. Therefore the metric is still the Minkowski one

$$\left(g_{\mathrm{FIX}}^{\mathrm{Sch}}\right)_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

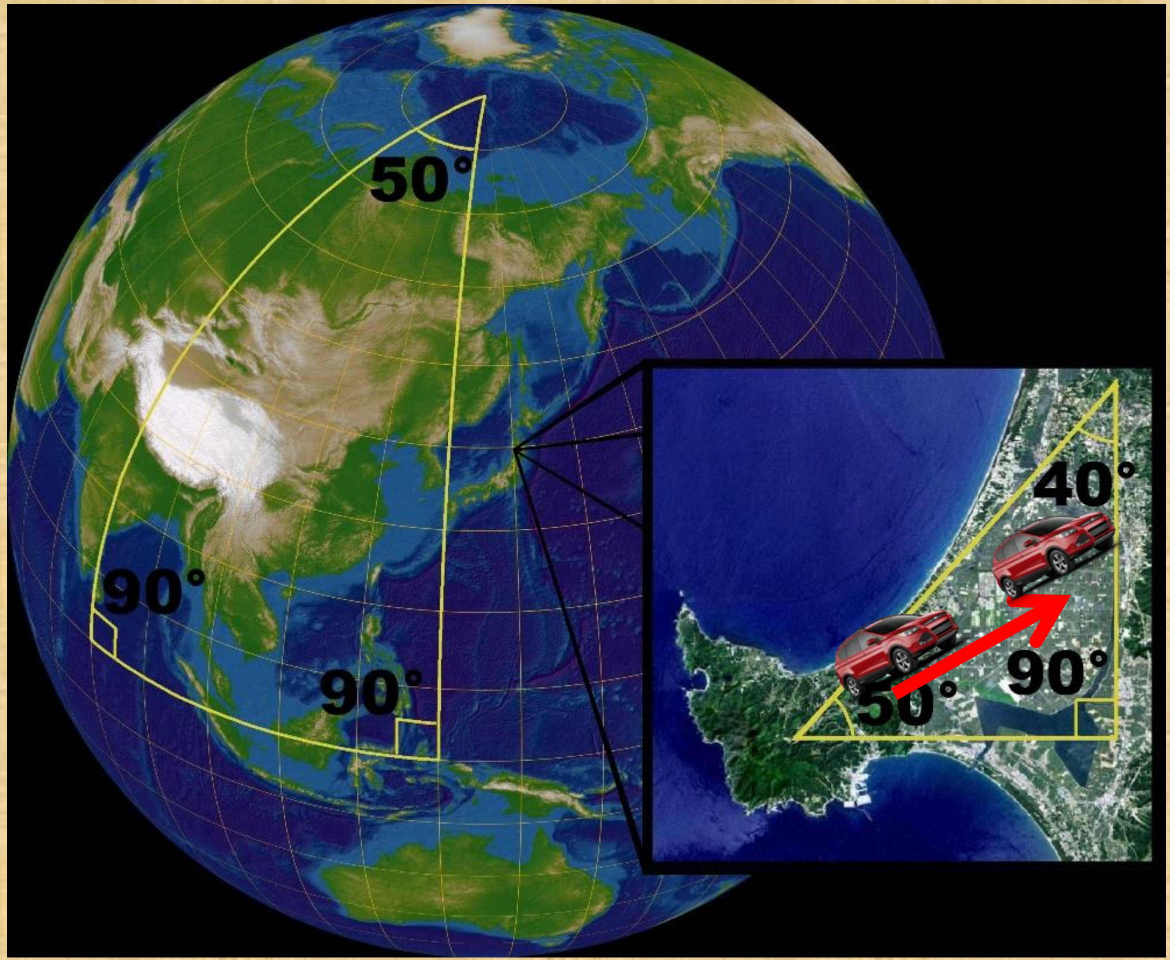
but now the 4-velocity of objects become

$$(U_{\mathrm{FIX}})^{\alpha} = \begin{pmatrix} \gamma c \\ \gamma V^{\hat{r}} \\ \gamma V^{\hat{\theta}} \\ \gamma V^{\hat{\phi}} \end{pmatrix}$$

which is obvious since the FIX and MOV frames are simply related by a Lorentz Transform.

It is a convenient frame for looking at motion of particles.
However, it is not unique, there is a different FIX frame for every point around the black hole.
This also means that time flows differently in different frames.

Schwarzschild-Hilbert frame (SH)



This is a global coordinate, so it does not have the problems in the FIX frame, there is a unique time coordinate and a single (r, θ, ϕ) system.

For this coordinate, the metric is the one we presented earlier

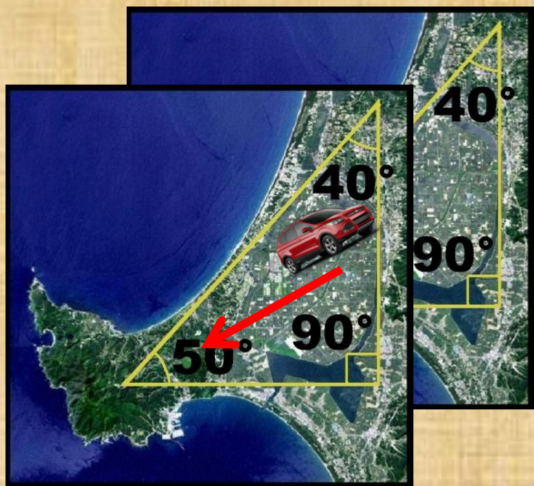
$$\left(g_{\mathrm{SH}}^{\mathrm{Sch}}\right)_{\alpha\beta} = \begin{pmatrix} -c^2\left(1-\frac{r_s}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\left(1-\frac{r_s}{r}\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

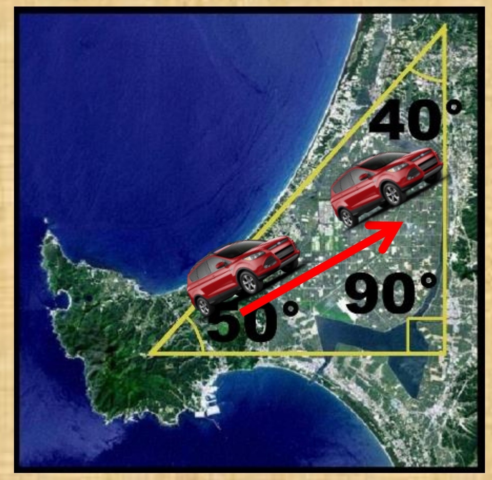
However, in such a case

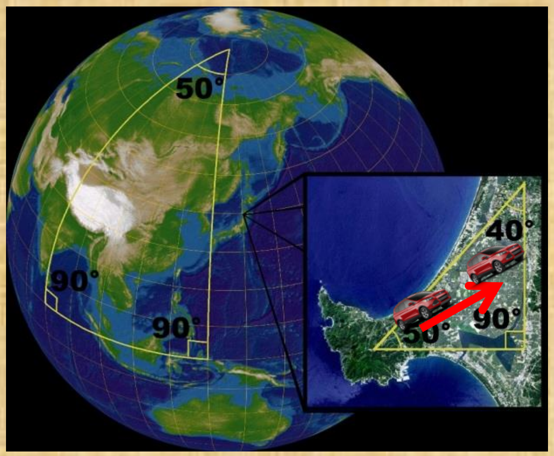
$$(U_{\mathrm{SH}})^{\alpha} = \begin{pmatrix} U^t \\ U^r \\ U^\theta \\ U^\phi \end{pmatrix}$$

is hard to interpret.

Which frame to use?







The usual solution to this dilemma is to employ a hybrid system that uses the best aspects of all three. Derivatives are expressed in the *global* (here SH) system, so solutions will be given as functions of (t, r, θ, ϕ). On the other hand, velocity, electric and magnetic field, and the charge/current four-vectors are expressed in the fixed local Lorentz (FIX) system, and we usually will use the standard three-velocity $\mathrm{V} = (V^{\hat{r}}, V^{\hat{\theta}}, V^{\hat{\phi}})$, with components in centimeters per second, instead of **U**. Finally, thermodynamic scalar quantities such as density and pressure are expressed in the moving-body system (MOV), where the familiar thermodynamic laws and equations of state are valid. This prescription, is sometimes called the "3+1 system", and is discussed in much more detail at the end of this chapter and in Chapter 9. Its great advantage is that variables are expressed in familiar terms and the equations look very similar to the laws of physics that we already know here on the earth.

But how is the 3+1 system accomplished in practice? The answer is through generalized Lorentz transformations. For example, let us consider the velocity. In

How to go from FIX to SH frame?

$$\left(g^{\mathrm{Sch}}_{\mathrm{SH}}\right)_{\alpha\beta}$$

$$= \begin{pmatrix} -c^2\left(1-\frac{r_s}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\left(1-\frac{r_s}{r}\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$\equiv \begin{pmatrix} g_{\mathrm{tt}} & 0 & 0 & 0 \\ 0 & g_{\mathrm{rr}} & 0 & 0 \\ 0 & 0 & g_{\theta\theta} & 0 \\ 0 & 0 & 0 & g_{\phi\phi} \end{pmatrix}$$

$$\left(g^{\mathrm{Sch}}_{\mathrm{FIX}}\right)_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g_{\alpha'\beta'} = \Lambda_{\alpha'}{}^{\alpha}\Lambda_{\beta'}{}^{\beta}g_{\alpha\beta}$$

$$\left(g^{\mathrm{Sch}}_{\mathrm{SH}}\right)_{\alpha'\beta'} = \Lambda_{\mathrm{SH}(\alpha')}{}^{\mathrm{FIX}(\alpha)}\Lambda_{\mathrm{SH}(\beta')}{}^{\mathrm{FIX}(\beta)}\left(g^{\mathrm{Sch}}_{\mathrm{FIX}}\right)_{\alpha\beta}$$

$$\begin{pmatrix} g_{\mathrm{tt}} & 0 & 0 & 0 \\ 0 & g_{\mathrm{rr}} & 0 & 0 \\ 0 & 0 & g_{\theta\theta} & 0 \\ 0 & 0 & 0 & g_{\phi\phi} \end{pmatrix} = \Lambda_{\mathrm{SH}(\alpha')}{}^{\mathrm{FIX}(\alpha)}\Lambda_{\mathrm{SH}(\beta')}{}^{\mathrm{FIX}(\beta)}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Generalized Lorentz Transform

$$\Lambda_{\mathrm{SH}}{}^{\mathrm{FIX}} = \begin{pmatrix} \sqrt{-g_{\mathrm{tt}}} & 0 & 0 & 0 \\ 0 & \sqrt{g_{\mathrm{rr}}} & 0 & 0 \\ 0 & 0 & \sqrt{g_{\theta\theta}} & 0 \\ 0 & 0 & 0 & \sqrt{g_{\phi\phi}} \end{pmatrix} = \begin{pmatrix} c\sqrt{1-\frac{r_s}{r}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-\frac{r_s}{r}}} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}$$

$$\Lambda_{\mathrm{FIX}}{}^{\mathrm{SH}} = \begin{pmatrix} \frac{1}{\sqrt{-g_{\mathrm{tt}}}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{g_{\mathrm{rr}}}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{g_{\theta\theta}}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{g_{\phi\phi}}} \end{pmatrix} = \begin{pmatrix} \frac{1}{c\sqrt{1-\frac{r_s}{r}}} & 0 & 0 & 0 \\ 0 & \sqrt{1-\frac{r_s}{r}} & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix}$$

Expressing the 4-velocity in SH coordinates

$$(U_{\mathrm{FIX}})^{\alpha} = \begin{pmatrix} \gamma \mathrm{c} \\ \gamma \mathrm{V}^{\hat{r}} \\ \gamma \mathrm{V}^{\hat{\theta}} \\ \gamma \mathrm{V}^{\hat{\phi}} \end{pmatrix} \quad (U_{\mathrm{SH}})^{\alpha} = \begin{pmatrix} U^{t} \\ U^{r} \\ U^{\theta} \\ U^{\phi} \end{pmatrix} \quad {\Lambda_{\mathrm{FIX}}}^{\mathrm{SH}} = \begin{pmatrix} \frac{1}{\sqrt{-g_{\mathrm{tt}}}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{g_{\mathrm{rr}}}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{g_{\theta\theta}}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{g_{\phi\phi}}} \end{pmatrix} = \begin{pmatrix} \frac{1}{c\sqrt{1-\frac{r_s}{r}}} & 0 & 0 & 0 \\ 0 & \sqrt{1-\frac{r_s}{r}} & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix}$$

$$(U_{\mathrm{SH}})^{\alpha'} = \begin{pmatrix} U^{t} \\ U^{r} \\ U^{\theta} \\ U^{\phi} \end{pmatrix} = {\Lambda_{\mathrm{FIX}(\alpha)}}^{\mathrm{SH}(\alpha')} (U_{\mathrm{FIX}})^{\alpha} = \begin{pmatrix} \frac{1}{c\sqrt{1-\frac{r_s}{r}}} & 0 & 0 & 0 \\ 0 & \sqrt{1-\frac{r_s}{r}} & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix} \begin{pmatrix} \gamma \mathrm{c} \\ \gamma \mathrm{V}^{\hat{r}} \\ \gamma \mathrm{V}^{\hat{\theta}} \\ \gamma \mathrm{V}^{\hat{\phi}} \end{pmatrix} = \begin{pmatrix} \frac{\gamma}{\sqrt{1-\frac{r_s}{r}}} \\ \sqrt{1-\frac{r_s}{r}}\,\gamma \mathrm{V}^{\hat{r}} \\ \frac{\gamma \mathrm{V}^{\hat{\theta}}}{r} \\ \frac{\gamma \mathrm{V}^{\hat{\phi}}}{r\sin\theta} \end{pmatrix}$$

This now becomes more convenient to use and interpret. ${U_{\mathrm{SH}}}^{\alpha}$ is the 4-velocity of the global frame and we write its components in terms of local frame parameters $\left(V^{\hat{r}}, V^{\hat{\theta}}, V^{\hat{\phi}}\right)$

Let's examine the 4-velocity

$$U_{SH}{}^{\alpha} = \begin{pmatrix} \dfrac{\gamma}{\sqrt{1-\dfrac{r_s}{r}}} \\ \sqrt{1-\dfrac{r_s}{r}}\,\gamma V^{\hat{r}} \\ \dfrac{\gamma V^{\hat{\theta}}}{r} \\ \dfrac{\gamma V^{\hat{\phi}}}{r\sin\theta} \end{pmatrix} = \begin{pmatrix} \dfrac{dt}{d\tau} \\ \dfrac{dr}{d\tau} \\ \dfrac{d\theta}{d\tau} \\ \dfrac{d\phi}{d\tau} \end{pmatrix}$$

We expect from our old idea of gravity that the velocity of objects should approach c as we get to the black hole, but if we check

$$\left(\frac{dr}{dt}\right)_{SH} = \frac{dr}{d\tau}\frac{d\tau}{dt} = \frac{U_{SH}{}^{r}}{U_{SH}{}^{t}} = \left(1-\frac{r_s}{r}\right)V^{\hat{r}}$$

Then when $r \to r_s$ $\left(\frac{dr}{dt}\right)_{SH} \to 0$!

Particles seen in the SH frame apparently are 'stuck' at the horizon and never get across it!

But particles should fall into black holes!

This is simply due to the Generalized Lorentz Transform.

What happened?

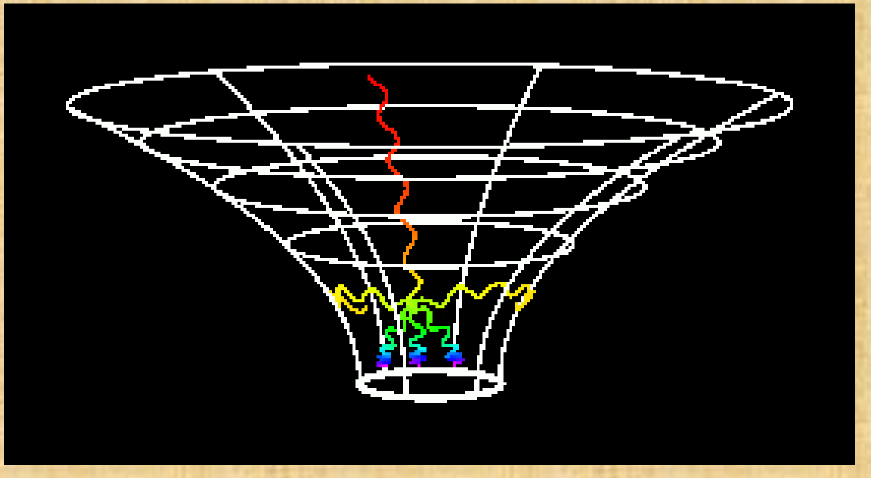
$$\left(\Lambda_{\mathrm{FIX}}{}^{\mathrm{SH}}\right)_{\mathrm{diag}} = \left(\frac{1}{c\sqrt{1-\frac{r_s}{r}}}, \sqrt{1-\frac{r_s}{r}}, \frac{1}{r}, \frac{1}{r\sin\theta}\right) \qquad \mathrm{dt}_{\mathrm{SH}} = \frac{\mathrm{dt}_{\mathrm{FIX}}}{\sqrt{1-\frac{r_s}{r}}}$$

Consider someone falling into a black hole, the local FIX frame observes the time of the person as $\mathrm{dt}_{\mathrm{FIX}}$, then, for a person sitting watching the BH very far away he would observe $\mathrm{dt}_{\mathrm{SH}}$.

Given that $\mathrm{dt}_{\mathrm{FIX}}$ should be finite, as $r \to r_s$, $\mathrm{dt}_{\mathrm{SH}} \to \infty$

It would take the far away observer infinite amount of time to watch the unfortunate person falling into the hole!

This also says that any photon sent out by the falling person would be infinitely redshifted.

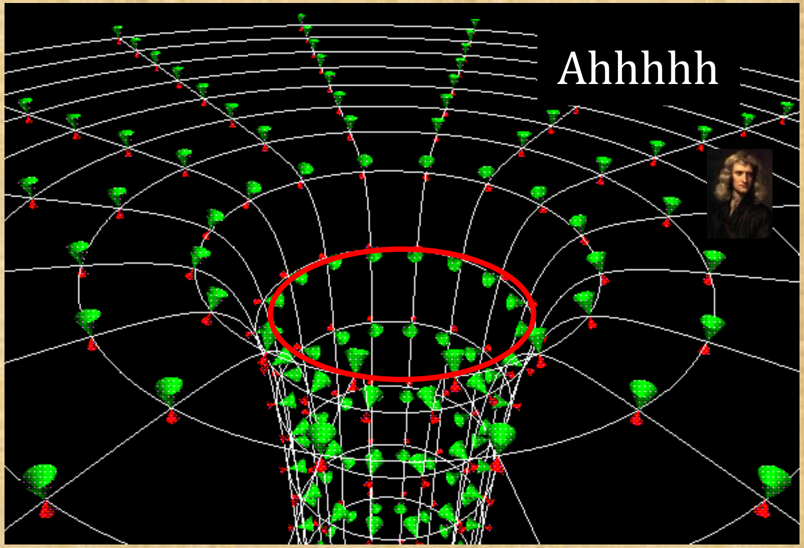


What happened?

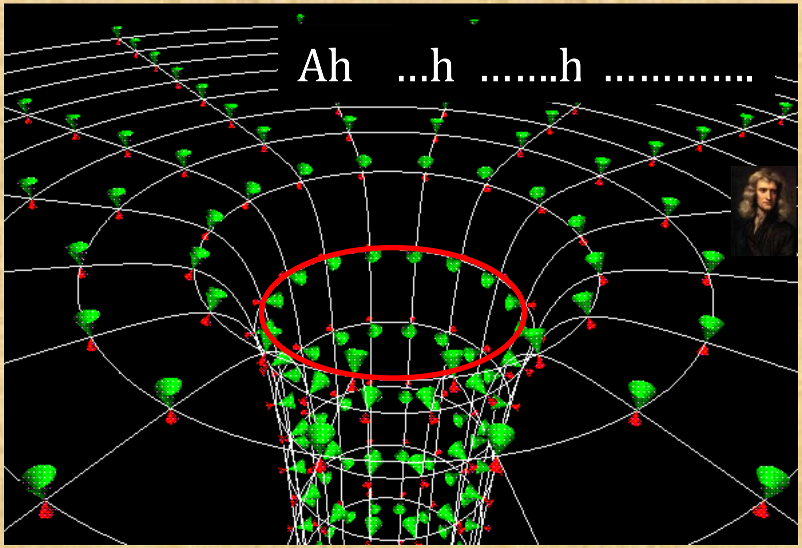
$$(\Lambda_{\mathrm{FIX}}{}^{\mathrm{SH}})_{\mathrm{diag}} = \left(\frac{1}{c\sqrt{1-\frac{r_s}{r}}}, \sqrt{1-\frac{r_s}{r}}, \frac{1}{r}, \frac{1}{r\sin\theta} \right) \qquad \mathrm{d}r_{\mathrm{SH}} = \mathrm{d}r_{\mathrm{FIX}}\sqrt{1-\frac{r_s}{r}}$$

Similarly, for finite $\mathrm{d}r_{\mathrm{FIX}}$, as $r \to r_s$, $\mathrm{d}r_{\mathrm{SH}} \to 0$!
No matter how much the person moves in some instant, a far away observer would observe him as stuck!

In the FIX frame



In the SH frame



Therefore combining $\mathrm{d}t_{\mathrm{SH}} = \frac{\mathrm{d}t_{\mathrm{FIX}}}{\sqrt{1-\frac{r_s}{r}}}$ and $\mathrm{d}r_{\mathrm{SH}} = \mathrm{d}r_{\mathrm{FIX}}\sqrt{1-\frac{r_s}{r}}$ it's obvious that the apparent velocity for an observer at infinity is zero !

The Equation of motion

In order to include gravity in the theory of relativity, Einstein reasoned that gravity must be a pseudo-force, arising not from another stress-energy component, but from the gradient operator itself (∇) in equation (6.121). In other words, because gravity occurs when matter is present, somehow matter must cause four-dimensional space to be curved, rather than flat. This curvature then gives rise to additional terms in the equations of motion that we interpret as the force of gravity. The addition

The equation of motion: $\left(\frac{d\vec{P}}{d\tau}\right)^{\alpha} = \left[\left(\vec{U}\cdot\vec{\nabla}\right)\vec{P}\right]^{\alpha} = U^{\beta}\left(\frac{\partial P^{\alpha}}{\partial \beta} + \Gamma^{\alpha}{}_{\mu\beta}P^{\mu}\right) = F^{\alpha}$

In general , the equation of motion expands to

$$F^{r} = U^{t}\left(\frac{\partial P^{r}}{\partial t} + \Gamma^{r}{}_{\mu t}P^{\mu}\right) + U^{r}\left(\frac{\partial P^{r}}{\partial r} + \Gamma^{r}{}_{\mu r}P^{\mu}\right) + U^{\theta}\left(\frac{\partial P^{r}}{\partial \theta} + \Gamma^{r}{}_{\mu\theta}P^{\mu}\right) + U^{\phi}\left(\frac{\partial P^{r}}{\partial \phi} + \Gamma^{r}{}_{\mu\phi}P^{\mu}\right)$$

Considering only radial motion, $U^{\theta} = U^{\phi} = 0$ and applying the relation between Christoffel symbols and the metric (we are now working in the SH coordinate),

$$U^{t}\left(\frac{\partial P^{r}}{\partial t} - \frac{1}{2}g^{\mathrm{rr}}\frac{\partial g_{\mathrm{tt}}}{\partial r}P^{t}\right) + U^{r}\left(\frac{\partial P^{r}}{\partial r} + \frac{1}{2}g^{\mathrm{rr}}\frac{\partial g_{\mathrm{rr}}}{\partial r}P^{r}\right) = 0$$

Hello Gravity !

$$U^t{}_{\mathrm{SH}}\left(\frac{\partial P^r{}_{\mathrm{SH}}}{\partial t_{\mathrm{SH}}}-\frac{1}{2}g^{\mathrm{rr}}{}_{\mathrm{SH}}\frac{\partial g_{\mathrm{tt}}{}^{\mathrm{SH}}}{\partial r_{\mathrm{SH}}}P^t{}_{\mathrm{SH}}\right)+U^r{}_{\mathrm{SH}}\left(\frac{\partial P^r{}_{\mathrm{SH}}}{\partial r_{\mathrm{SH}}}+\frac{1}{2}g^{\mathrm{rr}}{}_{\mathrm{SH}}\frac{\partial g_{\mathrm{rr}}{}^{\mathrm{SH}}}{\partial r_{\mathrm{SH}}}P^r{}_{\mathrm{SH}}\right)=0$$

Now comes the hidden trick used in the book… $P^r{}_{\mathrm{SH}}=\Lambda^{\mathrm{SH}}{}_{\mathrm{FIX}}P^r{}_{\mathrm{FIX}}=\frac{P^r{}_{\mathrm{FIX}}}{\sqrt{g_{\mathrm{rr}}}}$

$$\frac{\partial P^r{}_{\mathrm{SH}}}{\partial r_{\mathrm{SH}}}+\frac{1}{2}g^{\mathrm{rr}}{}_{\mathrm{SH}}\frac{\partial g_{\mathrm{rr}}{}^{\mathrm{SH}}}{\partial r_{\mathrm{SH}}}P^r{}_{\mathrm{SH}}=\frac{1}{\sqrt{g_{\mathrm{rr}}}}\frac{\partial P^r{}_{\mathrm{FIX}}}{\partial r_{\mathrm{SH}}}\cancel{-\frac{1}{2}\frac{P^r{}_{\mathrm{FIX}}}{(g_{\mathrm{rr}}{}^{\mathrm{SH}})^{1.5}}\frac{\partial g_{\mathrm{rr}}{}^{\mathrm{SH}}}{\partial r_{\mathrm{SH}}}+\left(\frac{1}{2}\frac{1}{g_{\mathrm{rr}}{}^{\mathrm{SH}}}\frac{\partial g_{\mathrm{rr}}{}^{\mathrm{SH}}}{\partial r_{\mathrm{SH}}}\right)\left(\frac{P^r{}_{\mathrm{FIX}}}{\sqrt{g_{\mathrm{rr}}}}\right)}$$

$$U^t{}_{\mathrm{SH}}\frac{\partial P^r{}_{\mathrm{FIX}}}{\partial t_{\mathrm{SH}}}+U^r{}_{\mathrm{SH}}\frac{\partial P^r{}_{\mathrm{FIX}}}{\partial r_{\mathrm{SH}}}-\underline{\frac{P^t{}_{\mathrm{SH}}U^t{}_{\mathrm{SH}}}{\sqrt{g_{\mathrm{rr}}}}\frac{1}{2}\frac{\partial g_{\mathrm{tt}}{}^{\mathrm{SH}}}{\partial r_{\mathrm{SH}}}}=0$$

Time dilation factor

It’s from the gradient operator!

$$\frac{\mathrm{d}P^r{}_{\mathrm{FIX}}}{\mathrm{d}\tau}=\frac{d\left(\gamma \mathrm{m}_0V^{\hat{r}}\right)}{\mathrm{d}\tau}=-\frac{\gamma}{\sqrt{1-\frac{\mathrm{rs}}{r}}}\left(\frac{GM(\gamma \mathrm{m}_0)}{r^2}\right)$$

General Relativistic Term!

Newtonian Gravity with relativistic mass

Conserved Quantities –1-forms are useful !

Henceforth, if unspecified, all the tensor/vector components are written in the SH coordinate

Again the equation of motion $\left(\frac{d\,\vec{P}}{\mathrm{d}\tau}\right)^{\alpha} = \left[\left(\vec{U}\cdot\vec{\nabla}\right)\vec{P}\right]^{\alpha} = U^{\beta}\left(\frac{\partial P^{\alpha}}{\partial\beta} + \Gamma^{\alpha}{}_{\mu\beta}P^{\mu}\right) = F^{\alpha}$

Considering the ϕ direction,

$$U^{t}\frac{\partial P^{\phi}}{\partial t} + U^{r}\left(\frac{\partial P^{\phi}}{\partial r} + \Gamma^{\phi}{}_{\phi r}P^{\phi}\right) + U^{\theta}\left(\frac{\partial P^{\phi}}{\partial\theta} + \Gamma^{\phi}{}_{\phi\theta}P^{\phi}\right) + U^{\phi}\left(\frac{\partial P^{\phi}}{\partial\phi} + \Gamma^{\phi}{}_{r\phi}P^{r} + \Gamma^{\phi}{}_{\theta\phi}P^{\theta}\right) = 0$$

Replacing in the definitions of the Christoffel symbols,

$$U^{t}\frac{\partial P^{\phi}}{\partial t} + U^{r}\frac{\partial P^{\phi}}{\partial r} + U^{\theta}\frac{\partial P^{\phi}}{\partial\theta} + U^{\phi}\frac{\partial P^{\phi}}{\partial\phi} + U^{r}P^{\phi}\frac{2}{r} + U^{\phi}P^{\theta}\frac{2\cos\theta}{\sin\theta} = 0$$

Finally, we get, $\frac{d\left(P^{\phi}r^{2}\sin^{2}\theta\right)}{\mathrm{d}\tau} = \frac{\mathrm{d}p_{\phi}}{\mathrm{d}\tau} \equiv \frac{\mathrm{dL}}{\mathrm{d}\tau} = 0 \qquad p_{\phi} = \gamma m_{0}V^{\hat{\phi}}r\sin\theta$

The angular momentum p_{ϕ} of a particle is conserved along the trajectory!

Conserved Quantities –1-forms are useful !

Similarly, we can also find that for the energy,

$$\frac{\mathrm{dE}}{\mathrm{d\tau}} = 0 \qquad E = -p_t = \sqrt{1 - \frac{r_s}{r}}\gamma m_0 c^2$$

The energy p_t of a particle is also conserved along the trajectory!

This constant, E, is sometimes also called energy at infinity because as $r \to \infty$, this term goes to $\gamma m_0 c^2$.

However, since it is the same at any radius, we can use it to calculate $\gamma(r)$ hence the velocity (we will see this on the next slide).

A moving body is bound to the BH if $\mathrm{E} < m_0 c^2$ and unbound otherwise.

Free fall

On the last slide we mention that we can use E to calculated the Lorentz factor as a function of radial distance r, let’s now work it out.

$$E = -p_t = \sqrt{1 - \frac{r_s}{r}}\gamma m_0 c^2$$

Consider a particle falling toward a black hole stating from rest at infinity.

This means that $E_\infty = m_0 c^2 = E(r) = \sqrt{1 - \frac{r_s}{r}}\gamma m_0 c^2$

This gives us $\gamma = \frac{1}{\sqrt{1 - \left(V^{\hat{r}}/c\right)^2}}$

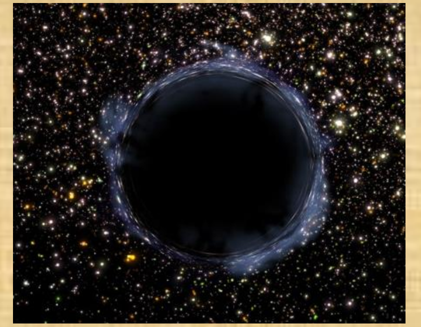
$$= \frac{1}{\sqrt{1 - \frac{\mathrm{rs}}{r}}} \rightarrow V^{\hat{r}}{}_{\mathrm{ff}} = -\sqrt{\frac{2GM}{r}}$$

At the event horizon, $V^{\hat{r}}{}_{\mathrm{ff}}(r_s) = -c$

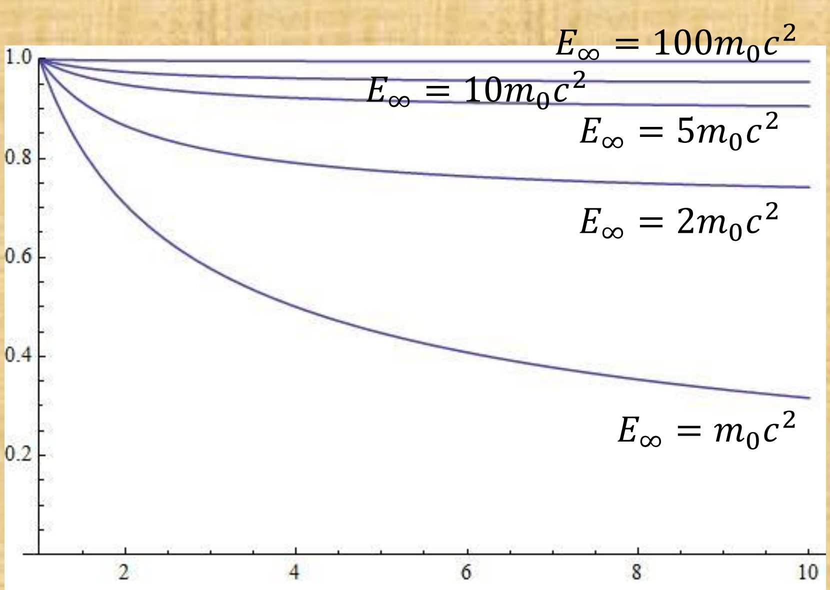
We see that if we drop something at infinity and assuming there is noting else in the universe between it and the BH, then it arrives at the BH at exactly the speed of light!

Chucking stuff directly at the BH

Now you might ask："Particles accelerate to c if we drop them off at infinity, what if we kick them into the BH starting from infinity?"



Special relativity tells us that we can't exceed c now matter what, so somehow the particle should still end up less than c even if we throw as hard as we can!



Let's consider a general case in which we don't specify energy at infinity, thus,

$$E_\infty = E(r) = \sqrt{1 - \frac{r_s}{r}}\gamma m_0 c^2$$

Solving this gives $V^{\hat{r}} = -c\sqrt{1 - \left(\frac{m_0 c^2}{E}\right)\left(1 - \frac{r_s}{r}\right)}$

Interestingly, no matter what E is, when $\mathrm{r} = r_s$, we always get $V^{\hat{r}} = -c$!

Orbits

Consider the simple cases of circular orbits, again using the equation of motion

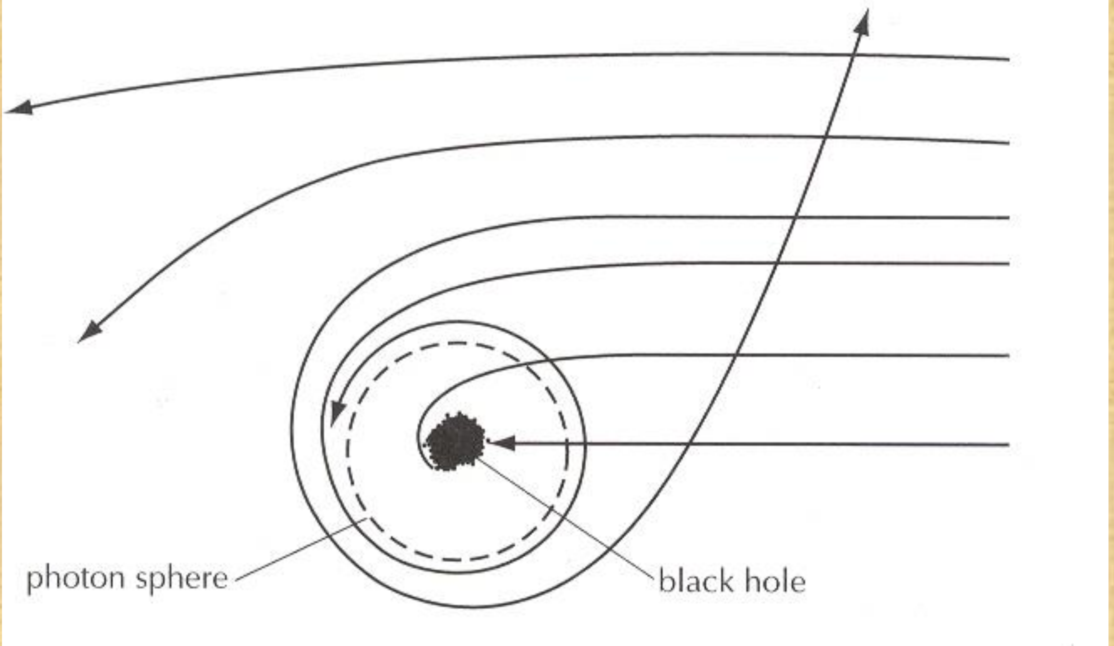
$$F^r = U^t\left(\frac{\partial P^r}{\partial t} + \Gamma^r{}_{\mu t}P^\mu\right) + U^r\left(\frac{\partial P^r}{\partial r} + \Gamma^r{}_{\mu r}P^\mu\right) + U^\theta\left(\frac{\partial P^r}{\partial \theta} + \Gamma^r{}_{\mu\theta}P^\mu\right) + U^\phi\left(\frac{\partial P^r}{\partial \phi} + \Gamma^r{}_{\mu\phi}P^\mu\right)$$

With some further reduction…

$$U^t\Gamma^r{}_{tt}P^t + U^\phi\Gamma^r{}_{\phi\phi}P^\mu = 0 \rightarrow (U^t)^2 g^{rr}\left(\frac{-1}{2}\frac{\partial g_{tt}}{\partial r}\right) = \left(U^\phi\right)^2 g^{rr}\left(\frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial r}\right)$$

We find that the orbital velocity in general is

$$V^{\hat{\phi}} = \sqrt{\frac{GM}{r - r_s}}$$



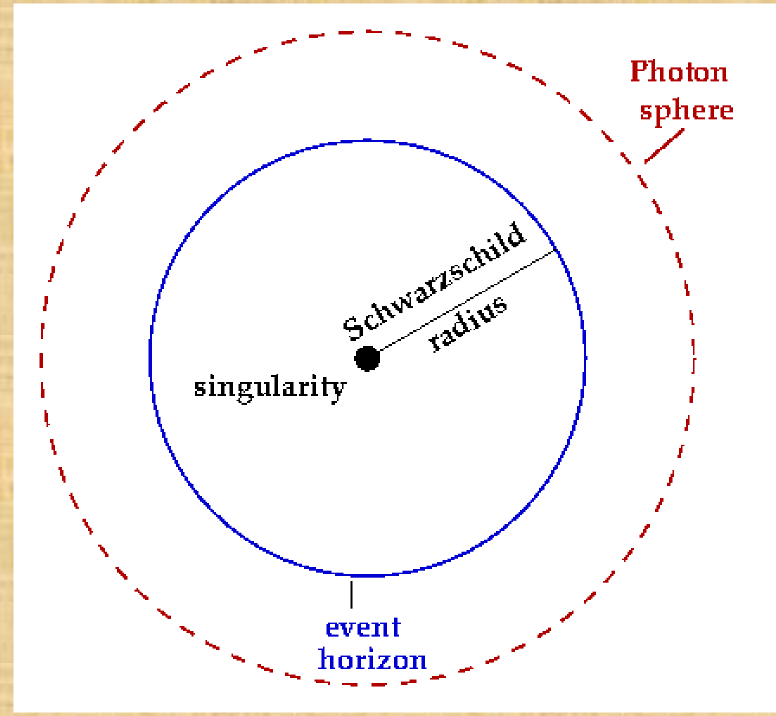
Photon Orbits

To find the orbital radius of photons, lets consider $V^{\hat{\phi}} = \sqrt{\frac{GM}{r - r_s}} = c$ case

This gives us $r_{\text{ph}} = \frac{3}{2} r_s$

i.e. for photons, the only place they can orbit the BH is at this radius.

However, we'll see later in a more general formulism (in Schutz) that this orbit is nowhere stable, if we accidently kick the photon a bit, it will either spiral into the BH or spiral out to infinity.



Finite mass particle orbits

For finite mass particles, we need to consider $V^{\hat{\phi}} = \sqrt{\frac{GM}{r-r_s}} < c$ case

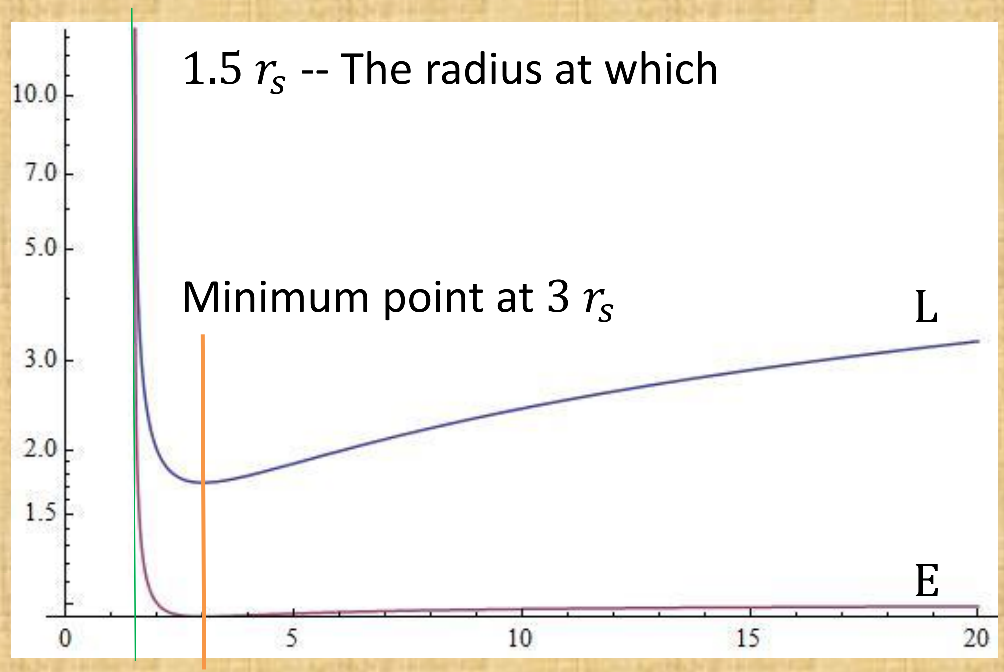
By definition of the Lorentz factor $\gamma_{\mathrm{orb}} = \frac{1}{\sqrt{1-\left(\frac{V_{\mathrm{orb}}}{c}\right)^2}} = \sqrt{\frac{r-r_s}{r-\frac{3}{2}r_s}}$

Solving for the energy and angular momentum,

$$L = p_\phi = \gamma m_0 V^{\hat{\phi}} r \sin\theta \qquad E = -p_t = \sqrt{1-\frac{r_s}{r}}\,\gamma m_0 c^2$$

$$L_{\mathrm{orb}} = \sqrt{\frac{r_s}{2r-3r_s}}\, r m_0 c^2$$

$$E_{\mathrm{orb}} = \frac{r-r_s}{\sqrt{r\left(r-\frac{3}{2}r_s\right)}} m_0 c^2$$



The ISCO

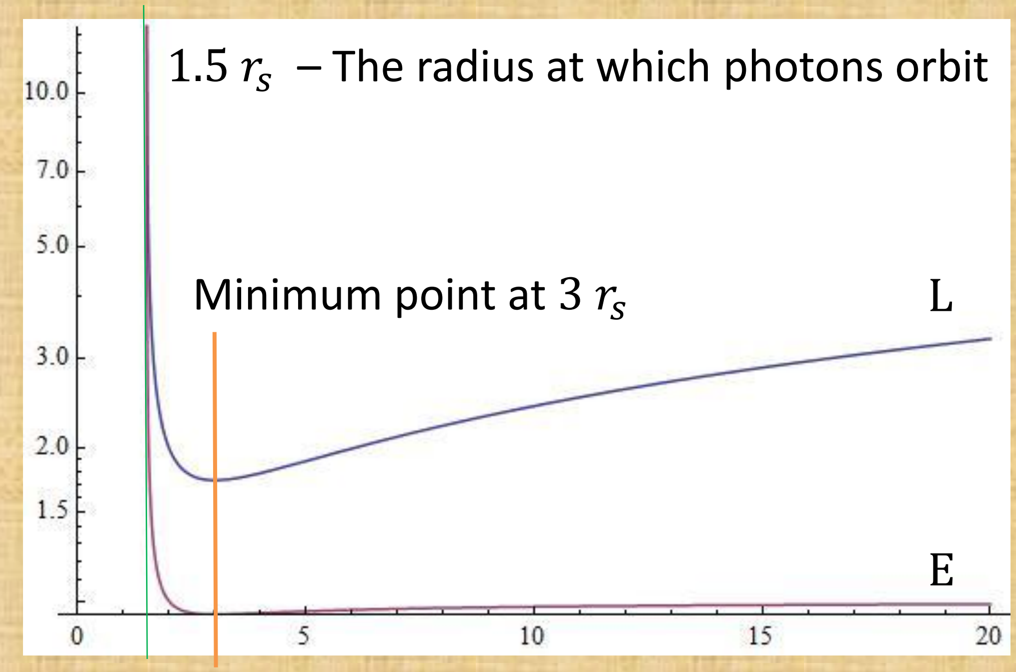
$$L_{\text{orb}} = \sqrt{\frac{r_s}{2r - 3r_s}} r m_0 c^2 \qquad E_{\text{orb}} = \frac{r - r_s}{\sqrt{r\left(r - \frac{3}{2} r_s\right)}} m_0 c^2$$

The minimum for both of these two curves happen at $\mathrm{r} = 3r_s$
This is commonly called the Innermost stable circular orbit for reasons we will see later.

At this radius, L and E are

$$L = \sqrt{3} r_s m_0 c \qquad E = \frac{2\sqrt{2}}{3} m_0 c$$

However, we will also find that the ISCO is more or less a ‘marginally stable’ orbit! If we accidently kick it a bit toward the black hole, it will just give up and fall in!



General discussion for particle motion

Previously, we have already found that we can calculate either free-fall or orbits by considering E or L respectively.

For a general consideration, it is more convenient if we write both of them in the same equation so we can discuss different the properties of the different orbits more clearly

$$P^2 = -{m_0}^2 c^2 = g_{\mathrm{tt}}(P^t)^2 + g_{\mathrm{rr}}(P^r)^2 + g_{\theta\theta}\left(P^\theta\right)^2 + g_{\phi\phi}\left(P^\phi\right)^2$$

$$E = -p_t = \sqrt{1 - \frac{r_s}{r}}\gamma m_0 c^2 \qquad L = p_\phi = \gamma m_0 V^{\hat{\phi}} r$$

For simplicity, we take $\theta = \pi/2$, so $P^\theta = 0$

$$-\frac{1}{c^2}\frac{1}{\left(1 - \frac{r_s}{r}\right)}E^2 + \frac{1}{\left(1 - \frac{r_s}{r}\right)}{m_0}^2\left(\frac{\mathrm{dr}}{\mathrm{d\tau}}\right)^2 + \frac{L^2}{r^2} = -{m_0}^2 c^2$$

$$\left(\frac{\mathrm{dr}}{c\mathrm{d\tau}}\right)^2 = \left(\frac{E}{m_0 c^2}\right)^2 - \left(1 - \frac{r_s}{r}\right)\left(1 + \frac{1}{r^2}\left(\frac{L}{m_0 c}\right)^2\right)$$

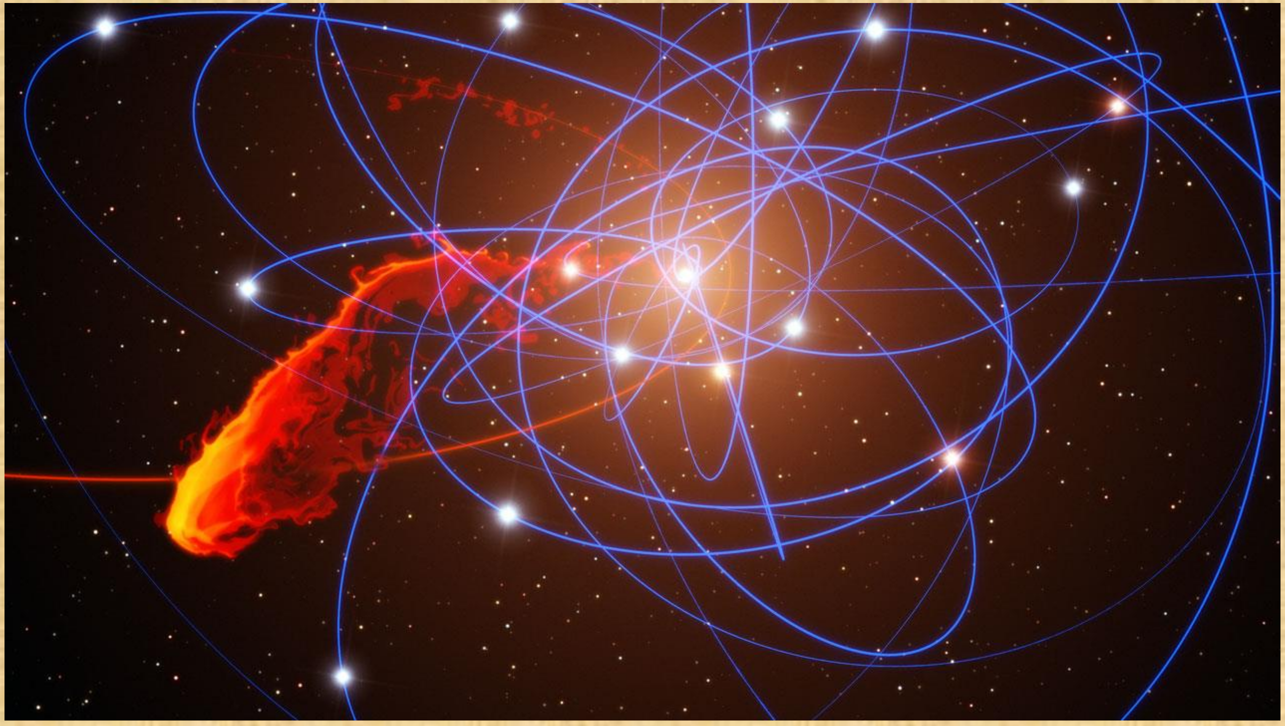
General discussion for particle motion

$$\left(\frac{\mathrm{dr}}{c\mathrm{d}\tau}\right)^2 = \left(\frac{E}{m_0c^2}\right)^2 - \left(1 - \frac{r_s}{r}\right)\left(1 + \frac{1}{r^2}\left(\frac{L}{m_0c}\right)^2\right)$$

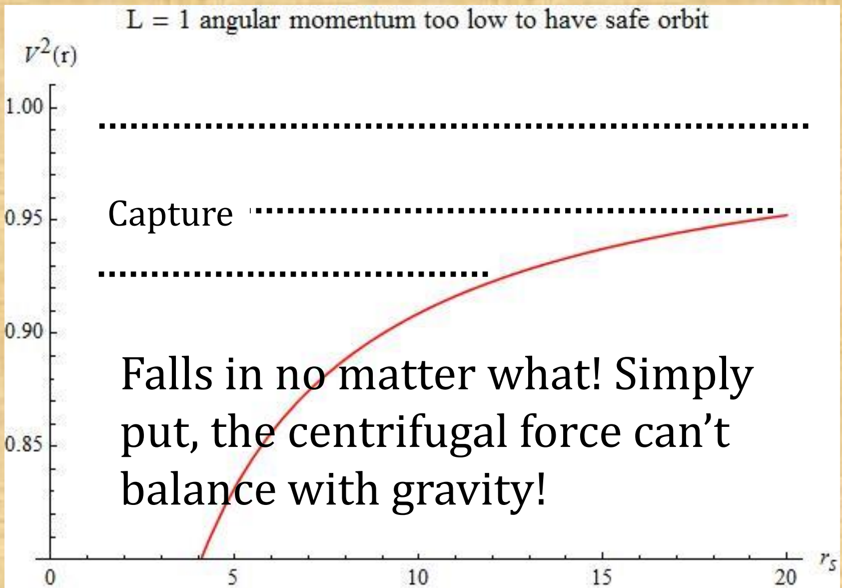
Remember that both E and L are constant of trajectory

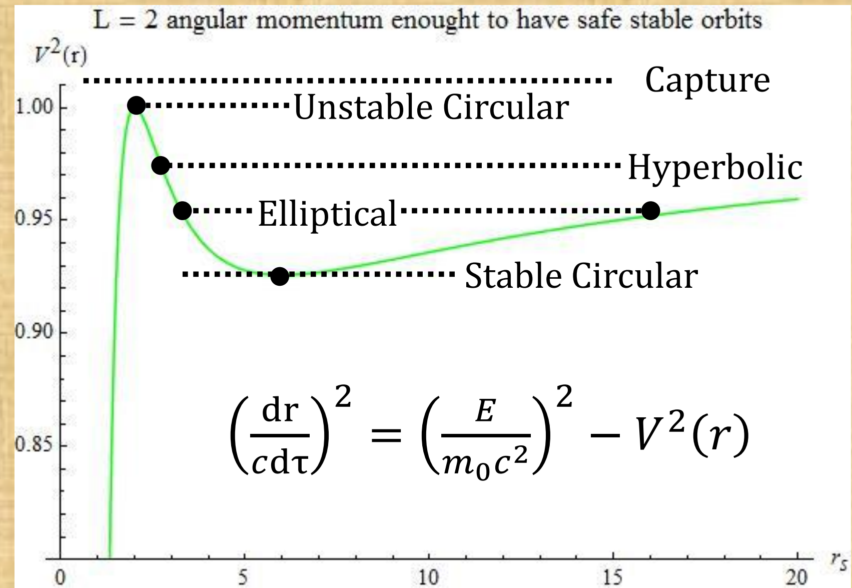
We can define the effective potential as $V^2(r) \equiv \left(1 - \frac{r_s}{r}\right)\left(1 + \frac{1}{r^2}\left(\frac{L}{m_0c}\right)^2\right)$

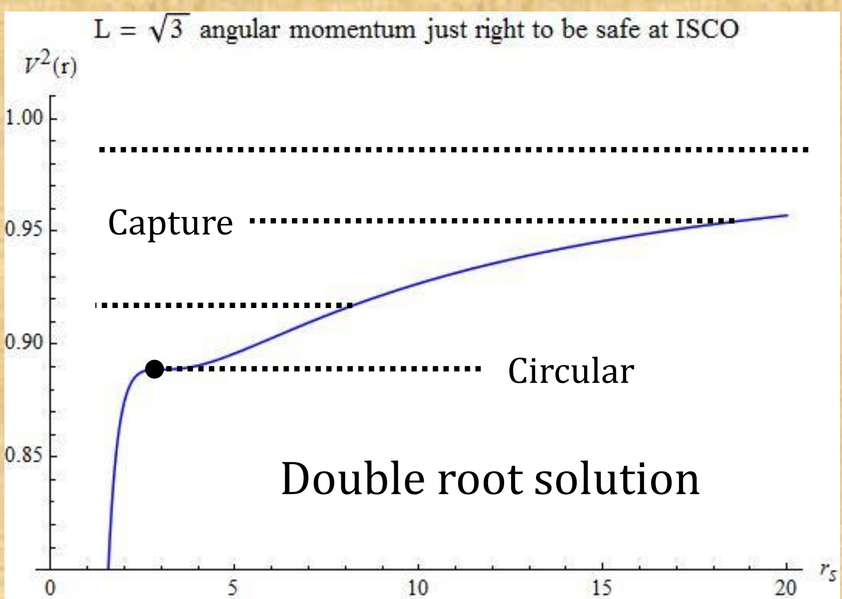
Then, $\left(\frac{\mathrm{dr}}{c\mathrm{d}\tau}\right)^2 = \left(\frac{E}{m_0c^2}\right)^2 - V^2(r)$ very much like the classical $E_k = E_{\mathrm{tot}} - V$!

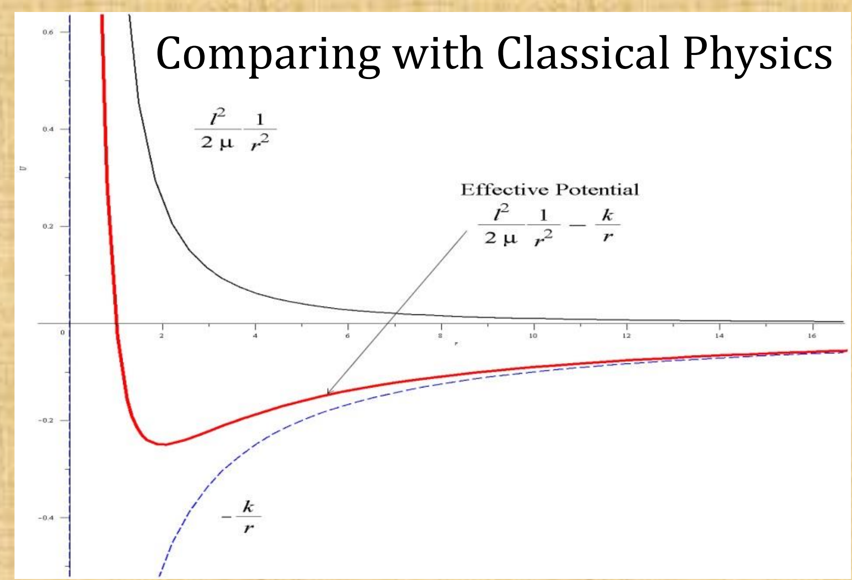


Behavior in different potentials $V^2(r) \equiv \left(1-\frac{r_s}{r}\right)\left(1+\frac{1}{r^2}\left(\frac{L}{m_0 c}\right)^2\right)$



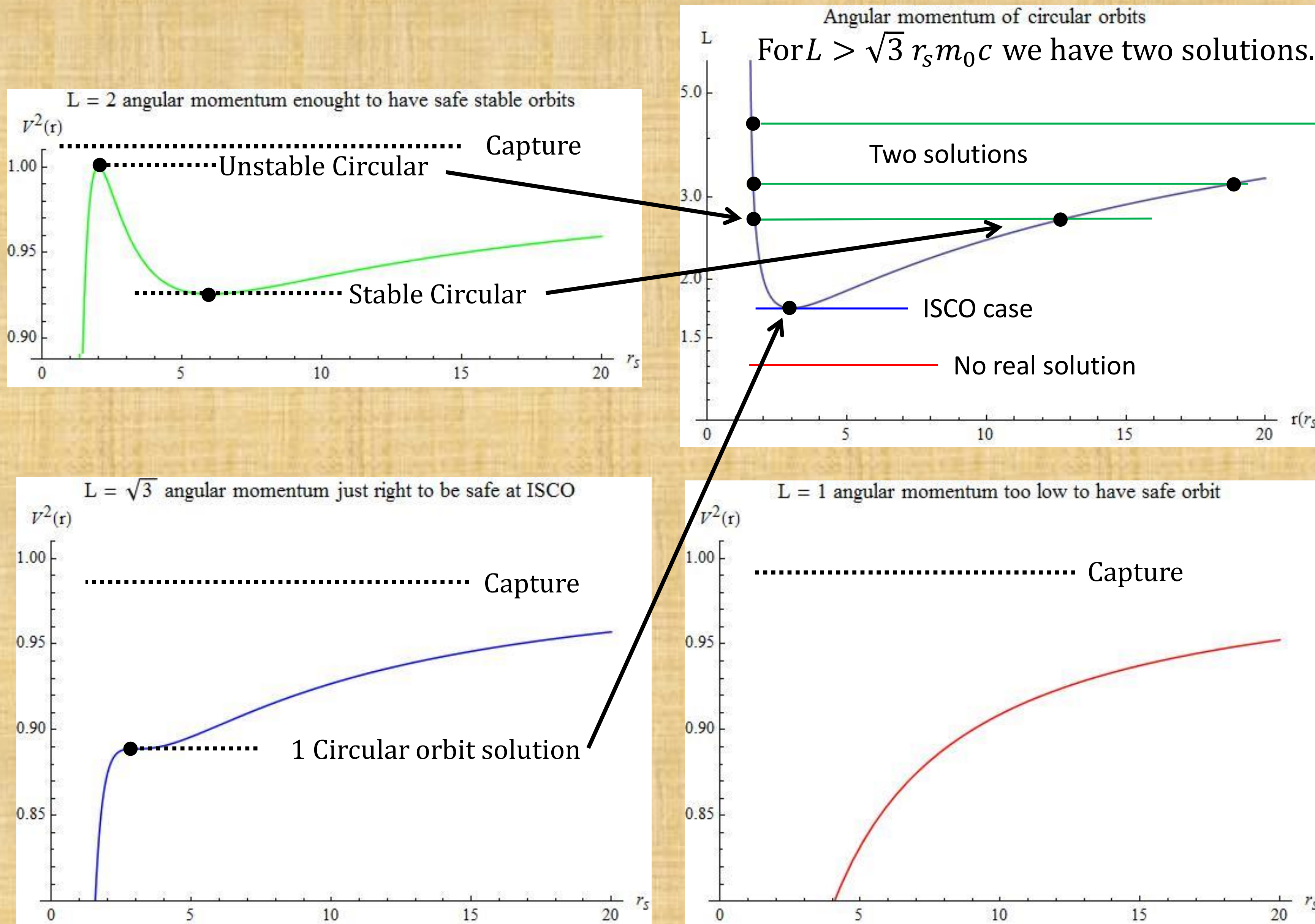




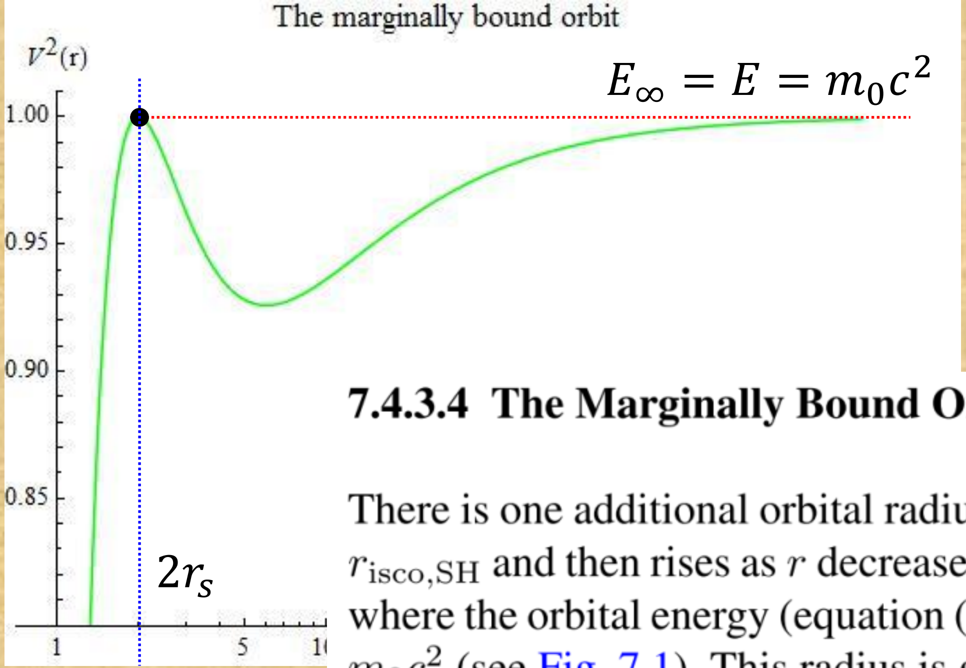


http://www2.warwick.ac.uk/fac/sci/physics/current/teach/module_home/px436/notes/lecture16.pdf

In relation to our previous analysis



The marginally bound orbit



7.4.3.4 The Marginally Bound Orbit

There is one additional orbital radius of interest. Because $\mathcal{E}_{\rm orb,SH}$ has a minimum at $r_{\rm isco,SH}$ and then rises as r decreases further, there must be radius $r_{\rm mb,SH} < r_{\rm isco,SH}$ where the orbital energy (equation (7.41)) rises back to the rest energy of the particle m_0c^2 (see Fig. 7.1). This radius is given by

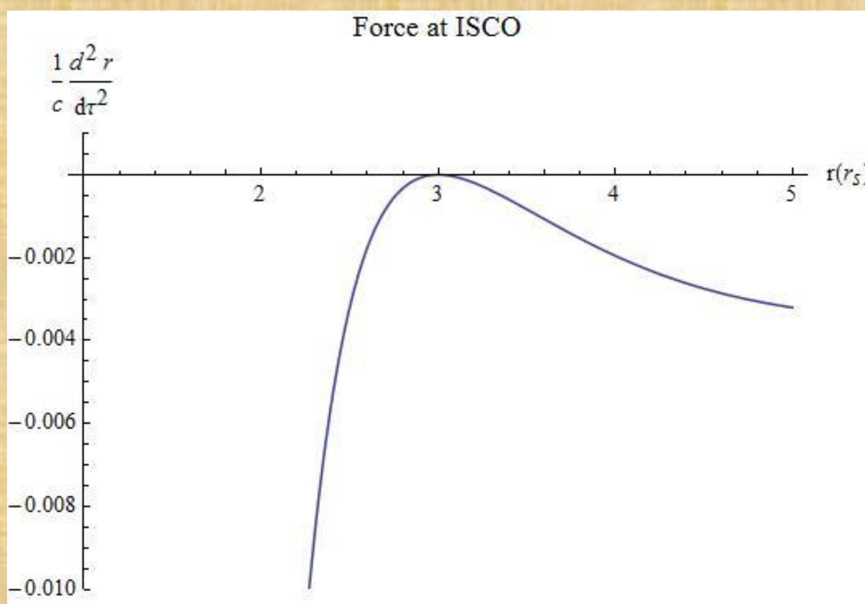
$$\boxed{r_{\rm mb,SH} = 4\frac{GM}{c^2}} \tag{7.44}$$

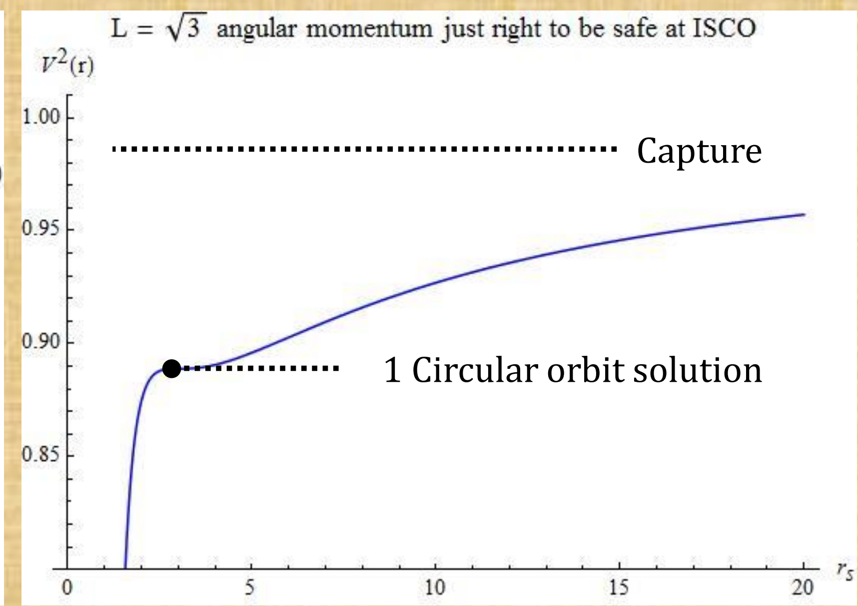
Between $r_{\rm isco,SH}$ and $r_{\rm mb,SH}$ orbits can exist, but they are unstable. Interior to $r_{\rm mb,SH}$ the binding energy is negative: more than the rest energy of the particle is needed just to keep it in a circular orbit, making these orbits very difficult to achieve. Like all orbits with $r < r_{\rm isco,SH}$, these orbits also are unstable. Generally, all particles in this region with finite mass will spiral into the black hole, unless a very powerful acceleration can move them into higher, safer orbits.

How stable is the ISCO?

$$\left(\frac{\mathrm{dr}}{c\mathrm{d}\tau}\right)^2 = \left(\frac{E}{m_0 c^2}\right)^2 - \left(1 - \frac{r_s}{r}\right)\left(1 + \frac{1}{r^2}\left(\frac{L}{m_0 c}\right)^2\right)$$

Taking the derivating w.r.t proper time on both sides, we get the force equation $\frac{1}{c}\frac{d^2 r}{\mathrm{d}\tau^2} = -\frac{1}{2}\frac{d}{\mathrm{dr}}\left[\left(1 - \frac{r_s}{r}\right)\left(1 + \frac{1}{r^2}\left(\frac{L}{m_0 c}\right)^2\right)\right]$ which is analogous to $\frac{d^2\vec{r}}{\mathrm{dt}^2} = \frac{\vec{F}}{m} = -\vec{\nabla} V$ in Classical Physics.

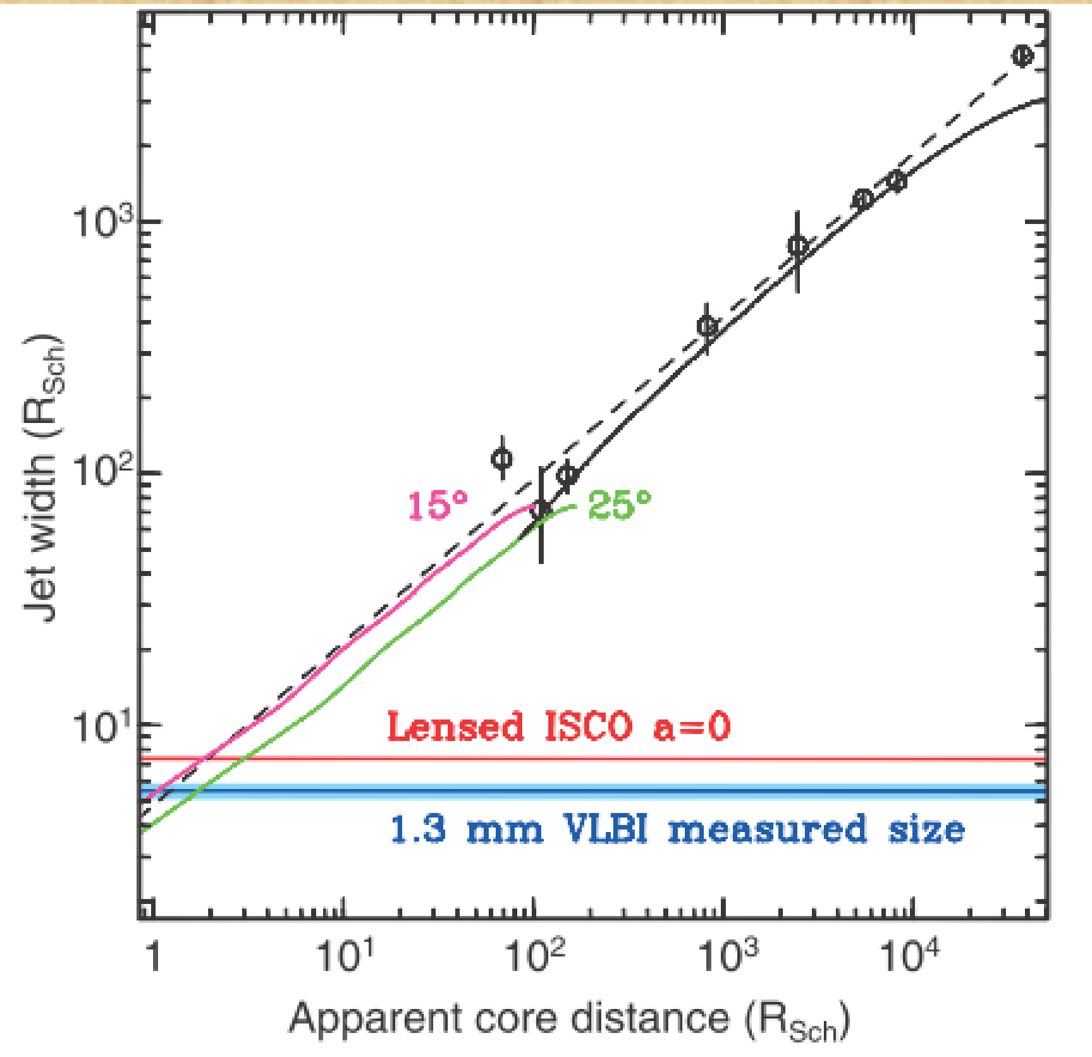




The ISCO is the double root solution, it is at the same time the stable and unstable circular orbit. Unfortunately for particles flying about the black hole, the result is simply that it is unstable! Any perturbation toward the black hole and the particle would have to say goodbye to the rest of the outside universe!

Observational evidence of the ISCO?

Resolving the Jet-Launch Region of the M87 Supermassive Black Hole ,
Science 338, 355 (2012)



The apparent singularity in the r coordinate of the Schwarzschild metric is a big problem, especially in computer simulations that simulate gas flows around, and into, black holes. While matter in *physical* space can flow easily into black holes, matter in simulations in Schwarzschild–Hilbert coordinates never reaches the horizon. Often numerical problems prevent the simulation from continuing past, say, a hundred or so Schwarzschild times

$$\tau_{\mathrm{S}} \equiv r_{\mathrm{S}}/c$$

However, while strange things do happen near the horizon in the Schwarzschild metric, a physical singularity is not one of them. The problem turns out to be all in the coordinates (particularly in time, actually) that are chosen to express the geometry, not the geometry itself.

We can see this easily by choosing a different time coordinate t', such that

$$dt' = dt + \frac{r_{\mathrm{S}}}{c\,(r - r_{\mathrm{S}})}\,dr$$

$$\left(g_{\mathrm{HP}}^{\mathrm{Sch}}\right)_{\alpha\beta} = \begin{pmatrix} -c^2\left(1-\dfrac{r_s}{r}\right) & \dfrac{r_s}{r}c & 0 & 0 \\ \dfrac{r_s}{r}c & 1+\dfrac{r_s}{r} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

The Horizon-Penetrating coordinates

$$\left(g^{\text{Sch}}_{\text{SH}}\right)_{\alpha\beta} = \begin{pmatrix} -c^2\left(1-\frac{r_s}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\left(1-\frac{r_s}{r}\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$\mathrm{d}t' = \mathrm{d}t + \frac{r_s}{c(r-r_s)}\mathrm{d}r$$

$$\Lambda^{\text{HP}(\alpha\prime)}{}_{\text{SH}(\alpha)} \equiv \begin{pmatrix} 1 & \frac{r_s}{c(r-r_s)} \\ 0 & 1 \end{pmatrix}$$

$$\left(g^{\text{Sch}}_{\text{HP}}\right)_{\alpha\beta} = \begin{pmatrix} -c^2\left(1-\frac{r_s}{r}\right) & \frac{r_s}{r}c & 0 & 0 \\ \frac{r_s}{r}c & 1+\frac{r_s}{r} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$\Lambda^{\text{HP}(\alpha\prime)}{}_{\text{SH}(\alpha)}\Lambda^{\text{HP}(\beta')}{}_{\text{SH}(\beta)}\left(g^{\text{Sch}}_{\text{HP}}\right)_{\alpha'\beta'} = \left(g^{\text{Sch}}_{\text{SH}}\right)_{\alpha\beta}$$

$$\begin{pmatrix} 1 & \frac{r_s}{c(r-r_s)} \\ 0 & 1 \end{pmatrix}^T \cdot \begin{pmatrix} -c^2\left(1-\frac{r_s}{r}\right) & \frac{r_s}{r}c \\ \frac{r_s}{r}c & 1+\frac{r_s}{r} \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{r_s}{c(r-r_s)} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{c^2(r-r_s)}{r} & 0 \\ 0 & \frac{r}{r-r_s} \end{pmatrix}$$

$g_{t\mu}$. The coordinate singularity at r_S now has “magically” gone away. However, the physical singularity at $r = 0$ remains, as does the horizon ($g_{t't'} = 0$) at $r = r_S$. This form of the metric also approaches the spherical-polar form of the Minkowski metric as r becomes much larger than r_S. But, for small r we now have a new, off-diagonal coefficient: $g_{t'r} = (r_S/r)c$ that has helped remove the coordinate singularity.

The most surprising thing about this transformation is that r is *still* the same radial coordinate as before; only the definition of time has changed. Yet the proper distance no longer diverges at $r = r_S$

$$s' = r\left(1 + \frac{r_S}{r}\right)^{1/2} + r_S \ln\left\{\left(\frac{r}{r_S}\right)^{1/2}\left[1 + \left(1 + \frac{r_S}{r}\right)^{1/2}\right]\right\}$$

which has the value $s' = \left[\sqrt{2} + \ln(1 + \sqrt{2})\right] r_S = 2.296\, r_S$ there. Instead, s' now measures the distance from the *physical* singularity at the black hole center out to the radius r, not from the Schwarzschild radius. Why is the proper distance different even though r is the same coordinate? The reason is that s and s' are measured on a “time slice” where $dt = 0$ or $dt' = 0$. These latter two different conditions, in two different coordinate systems, generate two different equations for the proper distance on a time slice. Numerical simulations that use this new Schwarzschild metric will have no trouble with matter flowing into the black hole.

The metric in equation (7.45) is an example where the coordinates “drift” with time. This occurs whenever any of the $g_{t'i}$ components are non-zero. In this case only $g_{t'r}$ is non-zero, so the drift must be in r only; it turns out that the drift here is inward. Had we chosen $dt' = dt - [r_S/c\,(r - r_S)]\,dr$, we would have obtained an *outgoing* system. See Section 7.7 for more details.